and

$$-\mathbf{a}\cdot\mathbf{b}\leq-\mathbf{a}\cdot\mathbf{b}^{P}+2(Q\Psi_t,HP\Psi^{P})+(Q\Psi_t,[H-E_{TN}+HG^{P}H]Q\Psi_t),\quad\text{(C4)}$$

where we have neglected the possible subtraction terms for simplicity. [Usually Eq. (C4) contains fewer subtractions than Eq. (C3). When there are infinite number of resonances below the Nth threshold, thus requiring an infinite number of subtractions in Eq. (C3), then Eq. (C3) would no longer be useful.]

Now, from the exact form of the solution Ψ given by

$$\begin{aligned}\Psi&=P\Psi^{P}+G^{P}HQ\Psi+Q\Psi\\&=P_{-N}\Psi^{P}+P_N\Psi^{P}+\mathcal{G}_{-N}HQ\Psi+\mathcal{G}_NHQ\Psi\\&\quad+G_{-N}H\mathcal{G}_NHQ\Psi+G_NH\mathcal{G}_{-N}HQ\Psi+Q\Psi,\end{aligned}\quad\text{(C5)}$$

it is possible to construct a function $Q_{-N}\tilde{\Psi}_t$ of the form

$$\begin{aligned}Q_{-N}\tilde{\Psi}_t&=P_N\Psi_t+Q\Psi_t\\&=P_N\Psi^{P}+\mathcal{G}_NHQ\Psi_t+G_NH\mathcal{G}_{-N}HQ\Psi_t+Q\Psi_t.\end{aligned}\quad\text{(C6)}$$

Substitution of Eq. (C6) into Eq. (C3) and simplifying the resulting expression, one obtains Eq. (C4). The calculation is tedious but perfectly straightforward, and the following relations prove useful:

$$G^{P}=\mathcal{G}_{-N}+G_{-N}H\mathcal{G}_N+G_NH\mathcal{G}_{-N}+\mathcal{G}_N,\quad\text{(C7)}$$

$$\mathbf{a}_{-N}\cdot\mathbf{b}_{-N}{}^{P-N}=a_{-N}\cdot b_{-N}{}^{P}+(P_{-N}\Psi^{P-N},HP_N\Psi^{P}),\quad\text{(C8)}$$

$$a_N\cdot b_{Nt}=a_N\cdot b_N{}^{P}+(P_N\Psi_t,HP_{-N}\Psi^{P})+(P_N\Psi^{P},[H-E_{TN}]P_N\Psi_t),\quad\text{(C9)}$$

$$\mathcal{G}_NHG_{-N}=G_NH\mathcal{G}_{-N},\quad\text{(C10)}$$

$$G_{-N}-\mathcal{G}_{-N}=-G_{-N}HG_NH\mathcal{G}_{-N}.\quad\text{(C11)}$$

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Measurement of Time Correlations for Quantum-Mechanical Systems*

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Measurements that are of limited accuracy, are incomplete, or require a finite time to make do not generally permit one to construct a wave function for describing a physical system. The use of such partial information to predict the results of subsequent measurements is studied here. There are several practical applications of this problem, including the use of the autocorrelation function for a particle counter in a scattering experiment.

I. INTRODUCTION

IT is customary in the pedagogical development of quantum mechanics and field theory to mention the limitations on correlated measurements of observables at different space-time points. Little attention has been given, however, to actual experiments for making such observations, or their usefulness. In this paper and in a subsequent one, we shall discuss both of these subjects from a general point of view and with particular applications to scattering processes.

This work is an outgrowth from a recent paper on the correlated counting rate of two detectors recording particles scattered from a target.[1] There it was shown that by such an observation both the magnitude *and* phase of a scattering amplitude can be determined. Such an observation of spatial correlations is only one of a much broader class of experiments to measure time and space-time correlations in a particle beam. For example, as we shall show in a subsequent paper, the time-dependent autocorrelation function for a single counter can provide information on the coherence of, say, a laser beam.[2] If a beam has been scattered, the autocorrelation function yields a measure of relaxation processes in the target.

In this paper we make some general comments on the theory of measurement for quantum-mechanical systems and illustrate the theory with some conceptually simple examples: (a) measurement of the spin of either one of two interacting particles at a time t_2 following the measurement of the spin of one of them at an earlier time t_1, and (b) the theory of intensity correlations of the Hanbury Brown-Twiss variety.

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[1] M. L. Goldberger, H. W. Lewis, and K. M. Watson, Phys. Rev. **132**, 2764 (1963).

[2] See, for example, C. H. Townes and R. Serber, *Quantum Electronics* (Columbia University Press, New York, 1960), p. 233.

II. MEASUREMENT OF TIME CORRELATIONS

We consider now some general questions pertaining to the theory of measurement when several observations are made in sequence on a given system. We imagine that the system being studied is described by a Hamiltonian H and that at time $t=0$ it has been prepared in a state $\psi(0)=\psi_0$. At any time $t>0$, the undisturbed system will develop in time according to the Schrödinger equation and be described by the state $\psi(t)$, where

$$\psi(t)=e^{-iHt}\psi_0. \tag{2.1}$$

Suppose now that at a particular time $t_1>0$ an observation is made of the state of the system. As is well known, we must interpret the results of this observation in a statistical sense.[3] We imagine that an ensemble of such systems has been prepared at the reference time $t=0$, each in state ψ_0,[4] and consider a set of measurements on the members of the ensemble at a later time t_1.

We begin to describe the results of the observation at t_1 in rather loose terms which will be made more precise as we proceed. The measurement of some set of observables for a given system will yield the result that the system at time t_1 is in a state λ corresponding to an eigenvector ω_λ (the particular states λ are characteristic of the observation of interest). If the measurement is repeated many times on different systems of the ensemble, we find the state λ occurring with probability $P(\lambda,t_1)$ given by

$$P(\lambda,t_1)=(\psi(t_1),e_\lambda\psi(t_1)), \tag{2.2}$$

where e_λ is the projection operator onto the state λ.

Since the Schrödinger equation is of first order in time derivatives, for times $t>t_1$, the wave function of those systems in the ensemble which were found to be in the state λ is

$$\begin{aligned}\psi_\lambda(t)&=C_\lambda \exp[-iH(t-t_1)]e_\lambda\psi(t_1)\\ &=\exp[-iH(t-t_1)]\omega_\lambda,\end{aligned} \tag{2.3}$$

where C_λ is a normalization constant. Following each subsequent observation which may be made on the system, new wave functions can be constructed in a similar manner.

This idealized description of a sequence of operations seems too restrictive to be of interest for most practical applications. For a variety of reasons, observations on any but the simplest systems will not determine a specific eigenstate. If the measurement involves a quantity that has a continuous spectrum, no precise determination is possible. Similarly if there is a degeneracy, one can say only that one has some linear combination of the degenerate eigenvectors (we return to this point below). This is related to the case where the observation is incomplete in the sense that only a fraction of all dynamical variables characterizing the system are ordinarily observed. Thus, the fact that a particle in a counter is within a certain macroscopic volume at a certain time may be determined—its momentum and spin orientation often not being observed. There is no difficulty in describing such partial observations on $\psi(t_1)$; one again merely constructs the projection operators for the given observation and lets these act on $\psi(t_1)$ to predict the results of the measurements over the ensemble. It is only when one wishes to use the results of an incomplete measurement at time t_1 to predict the outcome of subsequent observations at time t_2 that the theory becomes somewhat subtle.

It is conventional to say that a measurement takes place at a particular time t_1. In practice, any mechanical or electrical equipment has a finite response time; a signal recorded at time t_1 is a filtered response of impulses received at times earlier than t_1. It may not be possible to define precisely sequential observations on a system or even the intervals between such measurements. This difficulty is particularly acute when there is an inherent reason for requiring prolonged observation. For example, suppose we wish to first measure the energy and then the spin orientation with respect to an arbitrary axis of a particle with a magnetic moment placed in a magnetic field. If the time interval between the measurements were quite long, there would be no problem and the simple theory reviewed above would apply. The principle of complementarity, however, would preclude the simultaneous determination of both quantities. In the intermediate (and often more practical) case in which the second observation follows the first by an (effectively) finite time, the simple theory does not seem to apply.

Let us suppose that an *instantaneous* observation is made on a given system in the ensemble at a time t_1. The observation is that of the physical characteristic associated with the (Hermitian) operator J_1. Since in general J_1 does not represent a complete set of commuting operators for the system, observation of J_1 does not yield a precise specification of the state of the system. The set of operators that must be added to J_1 to make the customary complete set will be called O.

We assume that the operators (J_1,O) have a complete set of eigenvectors ω. The eigenvalues of J_1 will be called β_λ and we write

$$J_1\omega_{\lambda,\mu}=\beta_\lambda\omega_{\lambda,\mu}, \tag{2.4}$$

where the $\omega_{\lambda,\mu}$ are the eigenvectors corresponding to the eigenvalue β_λ; the index μ may run over a fixed set of values characterizing the degeneracy of the eigenvalue β_λ. It is a postulate of the theory of measurement that the observation made on the system will yield one of the eigenvalues β_λ as the numerical value of the physical property associated with J_1. (For ease of writing we

[3] J. Von Neuman, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).

[4] We shall later use a further ensemble average over initial states $\psi(0)$.

frequently refer to both the physical property and the operator associated with it simply as J_1. We also speak of that physical property and the operator as "having the value β_λ" when we mean in a more strict sense that the observed state is one for which J_1 has the eigenvalue β_λ.) Because the observation is incomplete, as a result of the observation we can only say that at time t_1, the system lies in the subspace of the projection operator E_λ given by

$$E_\lambda=\sum_\mu \omega_{\lambda,\mu}\omega_{\lambda,\mu}^\dagger, \tag{2.5}$$

where the sum runs over all μ satisfying Eq. (2.4) with λ fixed.

We now come to the fundamental question: To what extent do those portions of the wave function that are "not observed" remember their heritage? There are certainly circumstances in which all memory of the state prior to the measurement are eased, in which case we would say that the wave function $\psi_\lambda(t_1)$ is given by

$$\psi_\lambda(t_1)=\sum_\mu a_\mu\omega_{\lambda,\mu}, \tag{2.6}$$

where the a_μ are arbitrary, except for the normalization condition. Measurements that have such a drastic effect on a system seem to be rather poorly suited for meaningful sequential observations. To illustrate this, let us imagine that the wave function ψ describes energetic particles scattered by a small target. Ahead of the target is placed a counter to monitor the incident beam and to determine the precise time at which each particle is scattered. The scattered particles are detected by a counter telescope, so designed that each scattered particle entering the telescope is counted by each of a sequence of single counters in the telescope. Now, the observation of a count at time t_1 in the first detector of the telescope provides information concerning the magnitude and direction of the counted particle (since it traveled in a known time from the target to the telescope). This knowledge is incomplete, however, since the spin orientation is not observed. In spite of this, it is obvious that the available information permits one to predict the time of passage of the particle through the remaining counters of the telescope. Equation (2.6) would not permit us to make this prediction, however. By erasing the information that the particle had traveled to the telescope from the target, it is not possible even to predict that the direction of its velocity is such that it will pass through the remaining counters. We must conclude, therefore, that with arbitrary a_μ, Eq. (2.6) cannot take account of information that may be available from previous observations on the system.

Another possible way of specifying what one means by a partial measurement of a system might be to require that, since one is measuring J_1 which commutes with the other operators O, the expectation value of the O's should be the same before and after the measurement. This cannot be a generally correct conclusion. Consider the measurement of the square of the total angular-momentum operator J^2. It is clear that a measurement of the z component of the angular momentum J_z immediately afterward must give zero probability for eigenvalues of $J_z>j$, where $j(j+1)$ is the eigenvalue of J^2 given by the first measurement, whereas before the measurement this would not generally be the case. Thus, even though J_1 and O commute, the values taken by the O may be influenced by a measurement of J_1.

It is not clear that there is a universal answer to the question posed above—that is, to what extent does previous information persist after a new observation is made? We shall tentatively adopt here an answer to this question, which we state in a physically appealing form suggested to us by Wigner.[5] This is an assertion that, subject to whatever constraints are imposed by the observation of J_1 at time t_1, the wave function $\psi_\lambda(t_1)$ is that which provides maximum overlap,

$$\mathcal{O}=|(\psi_\lambda(t_1),\psi(t_1))|^2, \tag{2.7}$$

with the prior wave function $\psi(t_1)$. That is, we consider the statement that $\mathcal{O}$ be maximized with respect to variations of ψ_λ, subject to those constraints imposed by the observation at t_1, as a variational principle to determine $\psi_\lambda(t_1)$. This principle is interpreted as applying also to the case that J_1 represents a set of observables and to the case that the observation is of limited accuracy, from which it can be determined only that the system is in some domain of states.[6]

We shall call this the *principle of least interference*. To see how it is to be applied in a given situation, we return to the precise observation of a single observable J_1, as described by Eq. (2.4). Using the most general form (2.6) for $\psi_\lambda(t_1)$ compatible with the observed eigenvalue β_λ, we obtain from Eq. (2.7)

$$\mathcal{O}=\sum_{\mu,\nu} a_\mu^* a_\nu(\omega_{\lambda,\mu},\psi(t_1))(\psi(t_1),\omega_{\lambda,\nu}), \tag{2.8}$$

where

$$\sum_\mu |a_\mu|^2=1. \tag{2.9}$$

On varying, say, the a_μ^* to maximize $\mathcal{O}$, we obtain

$$0=\sum_\mu \delta a_\mu^*[(\sum_\nu a_\nu N_\nu)N_\mu^*-\eta a_\mu], \tag{2.10}$$

where η is a Lagrange multiplier introduced to satisfy Eq. (2.9) and

$$N_\nu\equiv(\psi(t_1),\omega_{\lambda,\nu}). \tag{2.11}$$

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[5] E. P. Wigner (private communication). The results of this paper had been obtained previously by a related argument.

[6] E. P. Wigner (Ref. 5) suggested the term "morally best" as applying to measurements for which $\mathcal{O}$ is a maximum. With poor experimental technique, one can presumably excessively disturb the system being studied and degrade his information (for example, one might accidentally blow up the laboratory).

From Eq. (2.10) we obtain

$$a_\mu = \frac{1}{\eta}(\sum_\nu a_\nu N_\nu) N_\mu^*,$$

and from Eq. (2.9) we obtain

$$\eta = |\sum_\nu a_\nu N_\nu|^2.$$

Thus, we have

$$a_\mu = N_\mu^* / \sum_\nu a_\nu^* N_\nu^* = (\omega_{\lambda,\mu}, \psi(t_1))(\sum_\nu a_\nu^* N_\nu^*)^{-1},$$

so that

$$\psi_\lambda(t_1) = \sum_\mu a_\mu \omega_{\lambda,\mu} = C_\lambda E_\lambda \psi(t_1). \tag{2.12}$$

Here E_λ is defined by Eq. (2.5), and C_λ is a normalization constant defined by the equation

$$|C_\lambda|^{-2} = (\psi(t_1), E_\lambda \psi(t_1)) \equiv P(\lambda, t_1), \tag{2.13}$$

where $P(\lambda,t_1)$ is just the probability that a system in the ensemble will have the value β_λ for J_1.

For times $t > t_1$, the wave function $\psi_\lambda(t)$ for the system has the form[7]

$$\psi_\lambda(t) = C_\lambda \exp[-iH(t-t_1)] E_\lambda \psi(t_1). \tag{2.14}$$

As an extreme illustration of the use of $\psi_\lambda(t)$ defined by Eq. (2.14) for subsequent measurements at times $t > t_1$, we imagine that ψ describes two completely independent systems, each in separated laboratories, isolated from each other. An observation made on one of these, call it our J_1, evidently does not influence the second system. This is precisely what our prescription, Eq. (2.14), says since E_λ acts as the identity operator on the variables of the second system.

In Sec. III we give some less extreme examples to show the plausibility of the prescription, Eq. (2.14), at least for an important class of measurements.

Now at a later time $t_2 > t_1$ we observe a quantity J_2 on those members of the original ensemble of systems for which J_1 was found to have the value β_λ at time t_1. Since J_2 does not necessarily commute with J_1 we must expect in general to supplement J_2 with a new set of commuting observables O' in order to have a complete specification of states. The eigenvalues of J_2 will be called b_l, and the eigenvectors $w_{l,m}$, so that

$$J_2 w_{l,m} = b_l w_{l,m}, \tag{2.15}$$

and the index m is the range of values characteristic of the degeneracy of b_l. The projection operator onto the subspace of the complete Hilbert space that corresponds to the eigenvalue b_l is

$$\mathcal{E}_l = \sum_m w_{l,m} w_{l,m}^\dagger, \tag{2.16}$$

with the sum extending over all states for which the eigenvalue of J_2 is b_l.

The probability that on the second observation at time t_2 the value of J_2 will be b_l is

$$P(l,t_2|\lambda,t_2) = (\psi_\lambda(t_2), \mathcal{E}_l \psi_\lambda(t_2)). \tag{2.17}$$

Evidently, the joint probability over the ensemble that J_1 has the value β_λ at time t_1 *and* that J_2 has the value b_l at time t_2 is

$$P(l,t_2;\lambda,t_1) = P(l,t_2|\lambda,t_1) P(\lambda,t_1), \tag{2.18}$$

where $P(\lambda,t_1)$ is given by the expectation value of E_λ in the state $\psi(t_1)$, Eq. (2.13). It is clear that similar considerations could be made for the observation of a third observable J_3 at a time $t_3 > t_2$

To write Eq. (2.18) out in detail, it is convenient to introduce Heisenberg operators,

$$J_2(\tau) = e^{iH\tau} J_2 e^{-iH\tau},$$

etc. Then, using Eqs. (2.14) and (2.17), we obtain

$$P(l,t_2|\lambda,t_1) = |C_\lambda|^2 (\psi(t_1), E_\lambda \mathcal{E}_l (t_2 - t_1) E_\lambda \psi(t_1)).$$

This and Eq. (2.13) permit us to write Eq. (2.18) as

$$\begin{aligned} P(l,t_2;\lambda,t_1) &= (\psi(t_1), E_\lambda \mathcal{E}_l (t_2-t_1) E_\lambda \psi(t_1)) \\ &= (\psi(0), E_\lambda(t_1) \mathcal{E}_l (t_2-t_1) E_\lambda(t_1) \psi(0)). \end{aligned} \tag{2.19}$$

The complete distribution function, Eq. (2.18), describing the values of J_1 at time t_1 and J_2 at time t_2 is generally difficult to measure. A simpler measurement is that of the average value of J_2, given that the first measurement of J_1 yielded the eigenvalue β_λ for the first observable, J_1. This implies that we calculate the mean value of J_2 at time t_2 for those members of the original ensemble which were found to have the value β_λ for J_1 at time t_1. Thus we compute

$$\begin{aligned} \langle J_2(t_2) \rangle_\lambda &\equiv (\psi_\lambda(t_2), J_2 \psi_\lambda(t_2)) \\ &= |C_\lambda|^2 (\psi(t_1), E_\lambda J_2 (t_2 - t_1) E_\lambda \psi(t_1)), \end{aligned} \tag{2.20}$$

where we have used the definition of $\psi_\lambda(t_1)$ given by Eq. (2.12). The correlation function for the two observations is then defined as the quantity

$$\begin{aligned} &\langle J_2(t_2) J_1(t_1) \rangle \\ &\quad \equiv \sum_{l,\lambda} b_l \beta_\lambda P(l,t_2;\lambda,t_1) \\ &\quad = \sum_\lambda \langle J_2(t_2) \rangle_\lambda \beta_\lambda P(\lambda,t_1) \\ &\quad = \sum_\lambda (\psi(t_1), E_\lambda J_2 (t_2 - t_1) \beta_\lambda E_\lambda \psi(t_1)) \\ &\quad = \sum_\lambda (\psi(t_1), E_\lambda J_2 (t_2 - t_1) J_1 E_\lambda \psi(t_1)) \\ &\quad = (\psi(t_1), [\sum_\lambda E_\lambda J_2 (t_2 - t_1) E_\lambda] J_1 \psi(t_1)) \\ &\quad = (\psi_0, [\sum_\lambda E_\lambda(t_1) J_2(t_2) E_\lambda(t_1)] J_1(t_1) \psi_0). \end{aligned} \tag{2.21}$$

[7] The form of $\psi_\lambda(t)$ given by Eq. (2.14) has been proposed by E. P. Wigner, Am. J. Phys. **31**, 6 (1963), and independently by ourselves in connection with the development given in Ref. 1. See also Albert Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1963), p. 199.

We have here used the fact that J_1 commutes with E_λ and have introduced the Heisenberg operators $E_\lambda(t_1)$ and $J_1(t_1)$. It should be noted that $J_1(t_1)$ may be placed on either side of the square bracket in the last two expressions of Eq. (2.21), as must be the case if $\langle J_2(t_2)J_2(t_1)\rangle$ is to be real.

In deriving the final form of Eq. (2.21) we have supposed that the sum on l extends over all possible values. The sum on λ may or may not extend over all values depending on the details of the observation.

The case in which $J_1(t_1)$ and $J_2(t_2)$ commute is evidently of special interest. We may then think of making a single observation of both $J_1(t_1)$ and $J_2(t_2)$. The correlation function (2.21) simplifies considerably in this case, particularly if we sum over all states λ. Since $J_2(t_2)$ and $E_\lambda(t_1)$ commute, we obtain just

$$\langle J_2(t_2)J_1(t_1)\rangle=(\psi(0),J_2(t_2)J_1(t_1)\psi(0)), \quad (2.22)$$

which is consistent with thinking of $J_2(t_2)J_1(t_1)$ as being a single observable.

We mentioned earlier that under many conditions the observations made on a system may be too inaccurate to determine specific eigenvalues of such observables as J_1 and J_2. In this case Eq. (2.21) must be modified. For example, if the only information obtained from the first measurement is that the value of J_1 is one of a set $s(\lambda)$ of eigenvalues β_λ, we introduce the projection operator

$$E_{s(\lambda)}=\sum_\lambda E_\lambda,$$

where the sum on λ extends over the set $s(\lambda)$. We again make the fundamental *assumption of least interference* and obtain [on following precisely the argument leading to Eq. (2.14)]

$$\psi_{s(\lambda)}(t)=C_\lambda' \exp[-iH(t-t_1)]E_{s(\lambda)}\psi(t_1). \quad (2.23)$$

The arguments leading from Eq. (2.14) to the final form of the correlated measurement $\langle J_2(t_2)J_1(t_1)\rangle$, Eq. (2.21), are now modified by replacing E_λ by $E_{s(\lambda)}$. We find in place of Eq. (2.21) the result

$$\langle J_2(t_2)J_1(t_1)\rangle = (\psi_0, [\sum_{s(\lambda)} E_{s(\lambda)}(t_1)J_2(t_2)E_{s(\lambda)}(t_1)]J_1(t_1)\psi_0), \quad (2.24)$$

where the sum runs over nonoverlapping sets $s(\lambda)$.

When the ensemble of systems is not in a pure state at $t=0$, but distributed over a set of states, we must perform a further ensemble average over the ψ_0 in Eq. (2.21) or (2.24). For example, Eq. (2.24) is then

$$\langle J_2(t_2)J_1(t_1)\rangle = \mathrm{Tr}\{\rho_0, [\sum_{s(\lambda)} E_{s(\lambda)}(t_1)J_2(t_2)E_{s(\lambda)}(t_1)]J_1(t_1)\}, \quad (2.25)$$

where ρ_0 is the density matrix for the ensemble at $t=0$. It is this latter form which we shall require in a subsequent paper for the description of scattering experiments.

Returning now to Eq. (2.21), let us imagine that J_2 was measured at a time t_2 prior to t_1, the time at which J_1 was measured. In this case the appropriate correlation function would evidently be

$$\langle J_1(t_1)J_2(t_2)\rangle = (\psi(t_2), [\sum_l \mathcal{E}_l J_1(t_1-t_2)\mathcal{E}_l]J_2\psi(t_2)) = (\psi_0, [\sum_l \mathcal{E}_l(t_2)J_1(t_1)\mathcal{E}_l(t_2)]J_2(t_2)\psi_0). \quad (2.26)$$

When we wish to consider both kinds of observations, it is convenient to introduce the time-ordered correlation function defined by

$$\langle T[J_2(t_2)J_1(t_1)]\rangle = \langle T[J_1(t_1)J_2(t_2)]\rangle = \langle J_2(t_2)J_1(t_1)\rangle, \quad \text{for} \quad t_2>t_1, = \langle J_1(t_1)J_2(t_2)\rangle, \quad \text{for} \quad t_1>t_2. \quad (2.27)$$

We note that when $J_1(t_1)$ and $J_2(t_2)$ commute, and the sum on l in Eq. (2.26) runs over all states, Eqs. (2.19) and (2.26) are identical. In this case neither observation interferes with the other, so the order in which these are made is irrelevant.

We have mentioned that because of the delayed transient response of all electrical and mechanical apparatus, the recorded observation will not correspond to an instant of time, but will be smeared over an interval. Our expressions for the correlated measurements must be corrected for this before they will correspond to the recorded observations. To do this, let us suppose that the basic observation of J_1 is relayed to the recording device via a linear transducer having the response function $L_1(\tau)$. By this we mean that an input signal $f(t')$ be registered as

$$\text{Output}=\int_{-\infty}^{t} dt' L_1(t-t')f(t').$$

The corresponding response function for J_2 is written as $L_2(\tau)$. We may therefore represent the two measurements by the "filtered operators"

$$\mathcal{J}_1(t_1)=\int_{-\infty}^{t_1} d\tau_1 L_1(t_1-\tau_1)J_1(\tau_1)$$

and (2.28)

$$\mathcal{J}_2(t_2)=\int_{-\infty}^{t_2} d\tau_2 L_2(t_2-\tau_2)J_2(\tau_2),$$

where the Heisenberg operators J_1 and J_2 continue to correspond to "instantaneous observables."

Application of the *principle of least interference* to the observation of these quantities may be less than straightforward. First, the technical problem of formulating the correct constraint associated with the observation of, say, $\mathcal{J}_1$, may be formidable. Second, when $|t_2-t_1|$ is less than the transient response time $\Delta\tau_d$ of

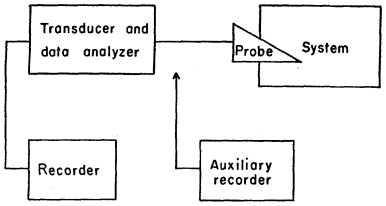

FIG. 1. Illustration of simultaneous use of narrow- and broad-band recording devices.

the measuring apparatus, so that τ_1 and τ_2 extend over overlapping intervals in Eq. (2.28), the mutual interferences of the two observations can be difficult to describe.

There are several special cases, however, for which observation of the two quantities (2.28) may be easily discussed. The simplest is that for which the response time $\Delta\tau_d$ of the detector, or measuring apparatus, is much less than the relaxation time of the system being studied. As long as $|t_2-t_1|$ is significantly larger than $\Delta\tau_d$, this is in effect an instantaneous observation to which we can apply the theory developed above. In particular, we may use Eqs. (2.14), (2.18), and (2.21) with the understanding that in these equations the operators J_1 and J_2 are to be replaced by $\mathcal{J}_1$ and $\mathcal{J}_2$. Because we have assumed that $\Delta\tau_d$ is smaller than the relaxation time of the system, we may take the projection operators E_λ and $\mathcal{E}_l$ to be unmodified by the transient response. Then the correlation function corresponding to Eq. (2.21) or (2.27) may be written in the concise form

$$G_{12}\equiv\langle\mathcal{J}_2(t_2)\mathcal{J}_1(t_1)\rangle$$
$$=\int_{-\infty}^{t_2}d\tau_2 L_2(t_2-\tau_2)\int_{-\infty}^{t_1}d\tau_1 L_1(t_1-\tau_1)$$
$$\times\langle T[J_2(\tau_2)J_1(\tau_1)]\rangle, \quad (2.29)$$

where the time-ordered correlation function was defined by Eq. (2.27).

A second special case to which the principle of least interference may be easily applied is the following. The fundamental observations on the system are made by probes, as in Fig. 1, which have response times very fast compared with relaxation times in S. This information is degraded, however, by narrow-bandwidth transducer recorders; so the detector response time $\Delta\tau_d$ may be larger than $|t_2-t_1|$ and system relaxation times. We assert that in spite of this we can again represent the measured correlation of the two observations by Eq. (2.29).

The reason for this is that we *might* have used fast recorders in parallel with the narrow-band devices and have recorded (say on a piece of paper) the quantity $\langle T[J_2(\tau_2)J_1(\tau_1)]\rangle$. The integrations implied in Eq. (2.29) could then be done numerically. The final result must agree with that recorded by the narrow-band device.

As a third and somewhat more complicated example, let us suppose that we are observing particles scattered by a specific target and that we are interested in observing relaxation processes in this target. The relaxation times in the target will be characterized by the interval $\Delta\tau_t$, while fluctuations in the particle beam itself will have a characteristic time $\Delta\tau_b$. We shall suppose that the experiment has been so designed that

$$\Delta\tau_b\ll\Delta\tau_d\ll\Delta\tau_t, \quad (2.30)$$

where $\Delta\tau_d$ again represents the response time of the detector. It is evident from the double inequality (2.30) that we can filter out the high-frequency beam fluctuations and observe only those of the target. The response functions L_1 and L_2 in Eq. (2.28) are assumed to describe the effect of these filters.

Equation (2.29) may be used to describe the correlation function for this experiment. To see this, we note from the first inequality (2.30) that during a single response time $\Delta\tau_d$ *many* scattered particles will be detected. Thus, random fluctuations in the particle fluxes will be small and for an ensemble average the $\mathcal{J}$'s and J's will be equivalent.

The fourth example that we consider is of some inherent complexity. We suppose that we are specifically studying fluctuations in a beam of scattered particles, but that we are unable to build a detector that can meet the condition $\Delta\tau_d\ll\Delta\tau_f$, where $\Delta\tau_f$ is a characteristic period of the fluctuations being studied. In this case we construct a composite detector, as illustrated in Fig. 2. At the center of the detector is placed a small scatterer s, having *known* scattering properties. Particle counters are mounted on the walls of the detector, which are a large distance from s. This "large distance" is defined by the condition that the particle-flux operators $J_2(\tau_2)$ and $J_1(\tau_1)$ at any pair of counters commute.

The beam of particles is first scattered at the target T, which is the system being studied. Some of these will enter the detector and be rescattered at s into counters on the walls. To analyze this experiment, we treat the beam, target, and scatterer s as a dynamical system described by the Schrödinger equation. The counters on the walls are considered to be the "classical" portion of the measuring apparatus, since their particle-current operators commute. With this arrangement we can study space-time correlations on a scale determined by the small scatterer s, but use only "classical" detectors.

To see this, let us first replace the instantaneous

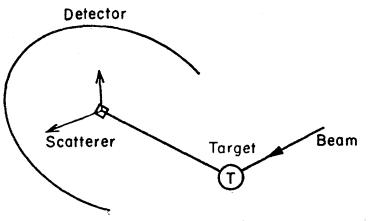

FIG. 2. Analysis of a scattering experiment.

particle-flux operators J_1 and J_2 by the filtered operators $\mathcal{J}_1(t_1)$ and $\mathcal{J}_2(t_2)$ in Eqs. (2.4), (2.5), (2.15), and (2.16). The filtered flux operators are supposed to represent the actual characteristics of the counters used in the detector. Because of the explicit time dependence, we shall now write the projection operators (2.5) and (2.15) as $E_\lambda(t_1)$ and $\mathcal{E}_l(t_2)$, respectively. Since $\mathcal{J}_1(t_1)$ and $\mathcal{J}_2(t_2)$ commute, we have

$$\begin{aligned}[\mathcal{J}_1(t_1),E_\lambda(t_1)]&=[\mathcal{J}_1(t_1),\mathcal{J}_l(t_2)]\\&=[\mathcal{J}_2(t_2),E_\lambda(t_1)]=[\mathcal{J}_2(t_2),\mathcal{E}_l(t_2)]\\&=[\mathcal{E}_l(t_2),E_\lambda(t_1)]=0.\end{aligned}\tag{2.31}$$

The wave function in the Heisenberg representation, following a specific observation of the two commuting quantities $\mathcal{J}_1(t_1)$ and $\mathcal{J}_2(t_2)$ is [according to Eq. (2.12)]

$$\psi_{l\lambda}(0)=C_{l\lambda}\mathcal{E}_l(t_2)E_\lambda(t_1)\psi_0,\tag{2.32}$$

since the two observations commute. Here

$$1/|C_{l\lambda}|^2=P(l,t_2;\lambda,t_1)=(\psi_0,E_\lambda(t_1)\mathcal{E}_l(t_2)\psi_0)\tag{2.33}$$

is the joint probability over the ensemble of systems of finding that $\mathcal{J}_1(t_1)$ and $\mathcal{J}_2(t_2)$ have the respective eigenvalues β_λ and b_l. The correlation function for this observation is [compare Eqs. (2.21) and (2.22)]

$$\begin{aligned}\langle\mathcal{J}_2(t_2)\mathcal{J}_1(t_1)\rangle&=\sum_{l,\lambda}b_l\beta_\lambda P(l,t_2;\lambda,t_1)\\&=\sum_{l,\lambda}(\psi(0),E_\lambda(t_1)\mathcal{E}_l(t_2)\mathcal{J}_2(t_2)\mathcal{J}_1(t_1)\psi(0))\\&=\sum_{\lambda}(\psi(0),\mathcal{J}_2(t_2)\mathcal{J}_1(t_1)E_\lambda(t_1)\psi(0)),\end{aligned}\tag{2.34}$$

where in the last step we have summed over all l. Now if we either sum also over all λ or if the operator J_1 satisfies the condition (which in fact is ordinarily satisfied for particle detectors[8])

$$J_1(t_1)E_\lambda(t_1)=J_1(t_1),\tag{2.35}$$

we obtain

$$\begin{aligned}G_{12}&=\langle\mathcal{J}_2(t_2)\mathcal{J}_1(t_1)\rangle\\&=\int_{-\infty}^{t_2}d\tau_2L_2(t-\tau_2)\int_{-\infty}^{t_1}d\tau_1L_1(t-\tau_1)\\&\quad\times(\psi(0),J_2(t_2)J_1(t_1)\psi(0)),\end{aligned}\tag{2.36}$$

which is equivalent to Eq. (2.29) for this case. In a subsequent paper we shall show how Eq. (2.36) may be used to study the target T of Fig. 2.

We now describe some simple illustrations of the general theory developed in this section.

III. EXAMPLE OF TWO COUPLED SPINS

We consider a system of two nonidentical spin-$\frac{1}{2}$ particles interacting through a spin-spin force described by the Hamiltonian

$$H=\alpha\sigma_1\cdot\sigma_2.\tag{3.1}$$

The parameter α may depend on the distance between the particles, but we shall not be concerned with this here. The eigenstates of H will be called χ_0 and $\chi^\nu(\nu=1,0,-1)$ where

$$\begin{aligned}H\chi_0&=(-3\alpha)\chi_0\equiv E_0\chi_0,\\\text{and}\qquad H\chi^\nu&=\alpha\chi^\nu\equiv E_1\chi^\nu.\end{aligned}\tag{3.2}$$

At any time t, the state of the system is described by

$$\psi(t)=e^{-iE_0t}a_0\chi_0+e^{-iE_1t}a^\nu\chi^\nu.\tag{3.3}$$

Projection operators for particle 1 to have its spin parallel or antiparallel to the z axis are

$$E_\pm^{(1)}=\tfrac{1}{2}(1\pm\sigma_z^{(1)}),\tag{3.4}$$

where $\sigma_z^{(1)}$ is the usual Pauli spin operator for particle 1. Suppose now we measure at a time t_1 the operator $E_+^{(1)}$ and then at a later time t_2 the operator $E_-^{(1)}$. In the notation of the previous section, we have $J_1=E_+^{(1)}$ and $J_2=E_-^{(1)}$. We have then for $\langle J_2(t_2)J_1(t_1)\rangle$ the result

$$\begin{aligned}\langle J_2(t_2)J_1(t_1)\rangle=(&e^{-iH(t_2-t_1)}E_+^{(1)}\psi(t_1),\\&E_-^{(1)}e^{-iH(t_2-t_1)}E_+^{(1)}\psi(t_1)).\end{aligned}\tag{3.5}$$

It is obvious by inspection that in the limit $t_2\to t_1$ we have $\langle J_2(t_2)J_1(t_1)\rangle\to 0$, which is physically reasonable, since the first measurement forced the spin of particle 1 to be up, and surely we must require this to be true just after the measurement. The complete expression for $\langle J_2(t_2)J_1(t_1)\rangle$ is easily evaluated and we find

$$\begin{aligned}\langle J_2(t_2)J_1(t_1)\rangle=\tfrac{1}{2}|&a_0\exp[-iE_0t_1]+a^0\exp[-iE_1t_1]|^2\\&\times\sin^2\tfrac{1}{2}[(E_1-E_0)(t_2-t_1)].\end{aligned}\tag{3.6}$$

Considerations based on this same physical system of two coupled spins serve to clarify further some of the remarks made in connection with our basic prescription for the wave function following a partial measurement, Eq. (2.12). In particular, the effect on operators not measured in the first instance (called O in Sec. II) may be studied by imagining first forcing the spin of particle 1 into a definite state, say spin up, so we take for J_1, as before $E_+^{(1)}$, and then ask for the expectation value over the ensemble of the z component of particle 2 at a later time. In such a case we have $J_2=\sigma_z^{(2)}$, and invite ourselves to consider

$$\begin{aligned}&\langle J_2(t_2)J_1(t_1)\rangle\\&\quad=(e^{-iH(t_2-t_1)}E_+^{(1)}\psi(t_1),\sigma_z^{(2)}e^{-iH(t_2-t_1)}E_+^{(1)}\psi(t_1)).\end{aligned}$$

We find, after an elementary calculation,

$$\begin{aligned}&\langle J_2(t_2)J_1(t_1)\rangle\\&\quad=|a^1|^2-\tfrac{1}{2}|a_0\exp[-iE_0t_1]+a^0\exp[-iE_1t_1]|^2\\&\qquad\times\cos(E_1-E_0)(t_2-t_1).\end{aligned}\tag{3.7}$$

[8] The information obtained from a count is that a particle is in the counter. If this is the case, $E_\lambda(t_1)$ is unity. If no particle is in the counter, $E_\lambda(t_1)$ and $J_1(t_1)$ both vanish. Thus, we obtain Eq. (2.35).

In the limit as $t_2 \to t_1$ this quantity is in general dramatically different from the expectation value of $\sigma_z^{(2)}$ in the state $\psi(t_1)$ before the measurement, namely,

$$\langle \sigma_z^{(2)} \rangle = (\psi(t_1), \sigma_z^{(2)} \psi(t_1)) = |a^1|^2 - |a^{-1}|^2 - 2\,\mathrm{Re}\{a_0^* a^0 \exp[i(E_0 - E_1)t_1]\}. \quad (3.8)$$

Thus, there is no question that $\langle O \rangle$ before the measurement of J_1 is equal to $\langle O \rangle$ afterward, even in situations that are not slightly pathological, as was our example in Sec. II of measurements of J^2 and J_z.

In spite of this example we maintain that our description of consecutive measurements is the reasonable one. To see this we consider now a situation where we start off our system of two spins in the triplet state for which $J_z = +1$, i.e., both spins up. We take for the operator J_1 the projection operator $E_{+\mathbf{n}}^{(1)}$, where

$$E_{\hat{n}}^{(1)} = \tfrac{1}{2}(1 + \boldsymbol{\sigma}^{(1)} \cdot \hat{n}), \quad (3.9)$$

and $\hat{n}$ is a unit vector $[\hat{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)]$ and for J_2 the projection operator $E_+^{(2)} = \tfrac{1}{2}(1 + \sigma_z^{(2)})$.

First, we find the probability $P(2\uparrow | 1\hat{n})$ that particle 2 has its spin up at time t_2 when it was found that at time t_1 particle 1 had its spin parallel to $\hat{n}$. According to Eq. (2.17) this is

$$P(2\uparrow | 1\hat{n}) = (\chi^1, E_{\hat{n}}^{(1)} E_+^{(2)} (t_2 - t_1) E_{\hat{n}}^{(1)} \chi^1)(\cos^2\tfrac{1}{2}\theta)^{-2},$$

which is easily evaluated to give

$$P(2\uparrow | 1\hat{n}) = \cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta \cos^2[\tfrac{1}{2}(E_1 - E_0)(t_2 - t_1)]. \quad (3.10)$$

In this case the orientation of spin 2 is not immediately affected by the measurement of the orientation of spin 1. That is, the probability of finding the second spin up just after the measurement is unity, as we would expect intuitively.

The correlation function (2.21) is found in this case to be

$$\langle J_2(t_2) J_1(t_1) \rangle = \cos^2\tfrac{1}{2}\theta [\cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta \cos^2\tfrac{1}{2}(E_1 - E_0)(t_2 - t_1)]. \quad (3.11)$$

As t_2 approaches t_1, we find $\cos^2\tfrac{1}{2}\theta$ as the value of the correlation function. This is simply the probability that the initial spin of particle 1 is to be found along the direction $\hat{n}$.

For the general state,

$$\psi(t_1) = a_0 \exp[-iE_0 t_1]\chi_0 + \exp[-iE_1 t_1](a^1\chi^1 + a^0\chi^0 + a^{-1}\chi^{-1}), \quad (3.12)$$

the correlation function $\langle E_+^{(2)}(t_2) E_{+\mathbf{n}}^{(1)}(t_1) \rangle$ is given by

$$\langle E_+^{(2)}(t_2) E_{1\mathbf{n}}^{(1)}(t_1) \rangle = |b^1|^2 + \tfrac{1}{2}|b_0 \exp[-iE_0 t_2] - b^0 \exp[-iE_1 t_2]|^2, \quad (3.13)$$

where

$$b^1 = \cos\frac{\theta}{2}\left\{\cos\frac{\theta}{2} a^1 + \sin\frac{\theta}{2}\frac{e^{-i\phi}}{\sqrt{2}} a^0 - \sin\frac{\theta}{2}\frac{e^{i\phi}}{\sqrt{2}} a_0 \exp[i(E_1 - E_0)t_1]\right\},$$

$$b^0 = \frac{1}{2}\left\{a^0 + \cos\theta a_0 \exp[i(E_1 - E_0)t_1] + \sin\theta\frac{e^{i\phi}}{\sqrt{2}} a^1 + \sin\theta\frac{e^{-i\phi}}{2} a^{-1}\right\}, \quad (3.14)$$

$$b_0 = \exp[i(E_0 - E_1)t_1]/2\left\{\cos\theta a^0 + a_0 \exp[i(E_1 - E_0)t_1] - \sin\theta\frac{e^{i\phi}}{\sqrt{2}} a^1 + \sin\theta\frac{e^{-i\phi}}{\sqrt{2}} a^{-1}\right\},$$

and the special case quoted above, Eq. (3.11), corresponds to the vanishing of all a's except a^1, which is unity. In general, therefore, the observation of the orientation of spin 1 with respect to the axis $\hat{n}$ has an instaneous effect on the state of the second spin. With the interpretation of an observation as making a selection among members of an ensemble, this is in no sense surprising.

IV. CORRELATED INTENSITY MEASUREMENTS

In this section we apply the principles introduced in Sec. II to a problem described by Fano[9] as an illustration of the Hanbury-Brown and Twiss intensity correlation.[10] Fano considers a physical system composed of four atoms—a, b, c, and d. At the beginning of the experiment, corresponding to time $t = 0$, the two identical atoms a and b are each in an excited state. During the course of the experiment, a and b undergo radiative transitions to their respective ground states. Fano calculates the probability that the radiation from a and b will photoionize both atoms c and d. Even in the limit of a small probability that either atom is ionized, the probability that both are ionized does not reduce to the

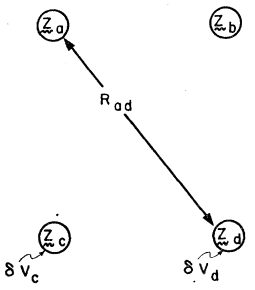

FIG. 3. Illustration of Fano's problem.

[9] U. Fano, Am. J. Phys. **29**, 539 (1961).
[10] R. Hanbury-Brown and R. Q. Twiss, Proc. Roy. Soc. (London) **A242**, 300 (1957); **A243**, 291 (1957).

"classical" product of the probabilities of ionizing the atoms individually.

The experiment is illustrated in Fig. 3. Atom a is located at position $\mathbf{z}_a$, b is at $\mathbf{z}_b$, etc. The distances R_{ab}, R_{ac}, etc. between atoms are assumed to be very large compared with the wavelength of the emitted radiation. In the interest of later geometrical simplifications we assume that R_{ab} and R_{cd} are much less than R_{ac}, etc. Also, to avoid irrelevant complication we ignore all details of the atomic processes of radiation and absorption, lumping these effects in unevaluated constants. (The reader who desires a detailed account should refer to Fano's paper.) Finally, we shall suppose the "radiation" emitted by either atom a or b to be a particle satisfying Bose-Einstein, or Fermi-Dirac statistics. The energy of such a particle will be called $\epsilon(k)$, corresponding to a momentum $\mathbf{k}$. If the particle is a photon the energy is $\epsilon(k)=kc$; for a particle of rest mass m we take $\epsilon(k)=k^2/2m$. The energy liberated by the radiative transition and converted into kinetic energy of the emitted particle is written as ϵ_0.

The wave function for the particle radiated by atom a is of the form[11]

$$\Phi_a(\mathbf{x},t)=\frac{i\gamma}{(2\pi)^2}\int\frac{d^3k\,\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{z}_a)]}{\epsilon(k)-\epsilon_0+i(\Gamma/2)}\times\{\exp[-i\epsilon(k)t]-\exp[-i(\epsilon_0-i(\Gamma/2))t]\}. \quad (4.1)$$

Here $\mathbf{x}$ is the coordinate of the radiated particle, $1/\Gamma$ is the lifetime of the atomic transition, and γ is a constant that is proportional to the probability amplitude for the transition. We have ignored the spin, if any, of the radiated particle in Eq. (4.1), which for our present argument is an inconsequential idealization.

In the limit that $|\mathbf{x}-\mathbf{z}_a|$ is large, we readily obtain from Eq. (4.1)

$$\begin{aligned}&\Phi_a(\mathbf{x},t)=0 \quad \text{for} \quad |\mathbf{x}-\mathbf{z}_a|>vt,\\ &\Phi_a(\mathbf{x},t)=-\frac{i\gamma p}{|\mathbf{x}-\mathbf{z}_a|\,v}\exp\{i[|\mathbf{x}-\mathbf{z}_a|\,p-\epsilon(p)t]\}\\ &\quad\times\exp\left[\frac{\Gamma}{2}\left(t-\frac{1}{v}|\mathbf{x}-\mathbf{z}_a|\right)\right], \text{ for } |\mathbf{x}-\mathbf{z}_a|<vt,\end{aligned} \quad (4.2)$$

where $\epsilon(p)=\epsilon_0$ and $v=d\epsilon(p)/dp$. An expression similar to Eq. (4.2) describes the radiation emitted by the atom b at $\mathbf{z}_b$.

To describe the radiation emitted by both a and b, we introduce the wave function[12]

$$\psi(t)=(1/\sqrt{2})[\Phi_a(\mathbf{x}_1,t)\Phi_b(\mathbf{x}_2,t)\pm\Phi_a(\mathbf{x}_2,t)\Phi_b(\mathbf{x}_1,t)], \quad (4.3)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the coordinates of the two emitted particles, and the + or − sign is to be used if the radiated particles satisfy Bose-Einstein or Fermi-Dirac statistics, respectively.

Following Fano's analysis, we might now calculate from Eq. (4.3) the probability that atom c absorbs a radiated particle at time t_1 and atom d absorbs the other particle at time t_2. To do this in detail, we should have to evaluate the transition amplitudes for absorption. We shall avoid this complication by doing a slightly different calculation.

To describe this, let us first imagine enclosing the atoms c and d in the respective very small volumes δV_c and δV_d. We shall then find the probability $P(d,t_2;\,c,t_1)$ that one of the radiated particles is in δV_c at time t_1 and that the other particle is in δV_d at a later time t_2. The probability that these particles are actually absorbed may be obtained from $P(d,t_2;\,c,t_1)$ on multiplication by the respective probabilities of absorption, given that that the particles are in δV_c and δV_d.

To evaluate $P(d,t_2;\,c,t_1)$ from Eq. (2.19) we must first construct the projection operators E_c and $\mathcal{E}_d=E_d$, which vanish unless a particle is in δV_c or δV_d, respectively. Then from Eq. (2.19) we obtain

$$P(d,t_2;\,c,t_1)=(\psi(0),E_c(t_1)E_d(t_2)E_c(t_1)\psi(0)). \quad (4.4)$$

The projection operators E_c and E_d may be constructed in terms of the projection operators

$$\begin{aligned}e_c(\mathbf{x}_1)&=\int_c d^3y\,\delta(\mathbf{x}_1-\mathbf{y}),\\ e_d(\mathbf{x}_1)&=\int_d d^3y\,\delta(\mathbf{x}_2-\mathbf{y}),\end{aligned} \quad (4.5)$$

etc., where the integrals extend over the volumes δV_c and δV_d, respectively. Then, we have

$$\begin{aligned}&E_c=e_c(\mathbf{x}_1)+e_c(\mathbf{x}_2)\\ \text{and}\\ &E_d=e_d(\mathbf{x}_1)+e_d(\mathbf{x}_2).\end{aligned} \quad (4.6)$$

To see that these are projection operators, we note that

$$E_c^2=e_c(\mathbf{x}_1)+e_c(\mathbf{x}_2)+2e_c(\mathbf{x}_1)e_c(\mathbf{x}_2).$$

Because we have assumed that the volumes δV_c and δV_d are very small, the term $2e_c(\mathbf{x}_1)e_c(\mathbf{x}_2)$ here is negligible when it appears in the integrand of Eq. (4.4).

In the Heisenberg representation, the operators (4.5) are of the form

$$\begin{aligned}e_c(\mathbf{x}_1,t)&\equiv\exp[iH_1t]e_c(\mathbf{x}_1)\exp[-iH_1t],\\ e_c(\mathbf{x}_2,t)&\equiv\exp[iH_2t]e_c(\mathbf{x}_2)\exp[-iH_2t],\end{aligned} \quad (4.7)$$

etc., where H_1 and H_2 are the respective Hamiltonians for the radiated particles 1 and 2. Since these particles

[11] See, for example, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, New York, 1964), Sec. 8.2, Eq. (8-119). The derivation of our Eq. (4.1) assumes $\Gamma\ll\epsilon$.

[12] Strictly speaking, the wave function (4.3) should also be symmetrized in the atomic coordinates and wave packets constructed to describe the localization of the atoms near $\mathbf{z}_a$ and $\mathbf{z}_b$. As long as these wave packets do not overlap this symmetrization is of course irrelevant, since it has no effect on our calculations.

do not interact, we have for any times t_1 and t_2,

$$\begin{aligned}[e_c(\mathbf{x}_1,t_1),\ e_c(\mathbf{x}_2,t_2)]&=0,\\ [e_c(\mathbf{x}_1,t_1),\ e_d(\mathbf{x}_2,t_2)]&=0,\end{aligned}\tag{4.8}$$

etc.

Using the commutation relations (4.8), we obtain

$$E_c(t_1)E_d(t_2)E_c(t_1)=e_c(\mathbf{x}_1,t)e_d(\mathbf{x}_2,t_2)+e_c(\mathbf{x}_2,t_1)e_d(\mathbf{x}_1,t_2)+\text{small terms}.\tag{4.9}$$

The "small terms," which we shall neglect, are of the form

$$e_c(\mathbf{x}_1,t_1)e_d(\mathbf{x}_1,t_2)e_c(\mathbf{x}_2,t_1),\quad \text{etc.},$$

and

$$e_c(\mathbf{x}_1,t_1)e_d(\mathbf{x}_1,t_2)e_c(\mathbf{x}_1,t_1),\quad \text{etc.}$$

Such terms all correspond to passage of a given particle through both volumes δV_c and δV_d. In the limit of small δV_c and δV_d, and assuming that these are not precisely in line with either radiating atom, these terms are negligible.[13]

Neglecting the "small terms," then, and using Eqs. (4.9) and (4.3) in Eq. (4.4), we immediately obtain

$$\begin{aligned}P(d,t_2;\,c,t_1)&=\int_c d^3y_1\int_d d^3y_2[\Phi_a^*(\mathbf{y}_1,t_1)\Phi_a(\mathbf{y}_1,t_1)\Phi_b^*(\mathbf{y}_2,t_2)\Phi_b(\mathbf{y}_2,t_2)\\ &\quad+\Phi_b^*(\mathbf{y}_1,t_1)\Phi_b(\mathbf{y}_1,t_1)\Phi_a^*(\mathbf{y}_2,t_2)\Phi_a(\mathbf{y}_2,t_2)\\ &\quad\pm\Phi_a^*(\mathbf{y}_1,t_1)\Phi_b(\mathbf{y}_1,t_1)\Phi_b^*(\mathbf{y}_2,t_2)\Phi_a(\mathbf{y}_2,t_2)\\ &\quad\pm\Phi_b^*(\mathbf{y}_1,t_1)\Phi_a(\mathbf{y}_1,t_1)\Phi_a^*(\mathbf{y}_2,t_2)\Phi_b(\mathbf{y}_2,t_2)].\end{aligned}\tag{4.10}$$

[13] Such terms would not appear in Fano's calculation, since his particles are absorbed and cannot subsequently reach the second term.

Since we have assumed that δV_c and δV_d are very small, we may evaluate this as

$$\begin{aligned}P(d,t_2;\,c,t_1)&=\delta V_c\delta V_d 2(\gamma p/R^2v)^2\\ &\quad\times\exp[-\Gamma(t_2-(R/v))]\exp[-\Gamma(t_1-(R/v))]\\ &\quad\times\{1\pm\cos[p(R_{ad}-R_{bd}+R_{bc}-R_{ac})]\}.\end{aligned}\tag{4.11}$$

Here we have set $R\approx R_{ad}\approx R_{bd}\approx R_{bc}\approx R_{ac}$ in all but the oscillating term.

Equation (4.11) corresponds to Fano's result. It illustrates the mutual interference of the two absorption processes, even in the limit of a vanishing probability [i.e., δV_c, $\delta V_d\to 0$] for absorption. This is closely related to the Hanbury-Brown and Twiss experiment[10] and to the scattering experiment proposed in Ref. 1. We note that expression (4.11) is symmetric in t_2 and t_1 and so is valid for all values of t_2 and t_1. This is a consequence of our use of very small volumes δV_c and δV_d.

In the second paper of this work we shall be concerned with practical applications of the phenomena studied here to scattering experiments. We shall, for example, describe corrections to Eq. (4.11) for finite sized volumes δV_c and δV_d, corresponding to counters used in a particle beam. We shall also discuss fluctuations and the autocorrelation function for a single-particle counter.

ACKNOWLEDGMENT

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